A unified framework for understanding and comparing dynamic wage and price setting models*

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Abstract

This paper argues that the cross-sectional approach to durations is essential to understand nominal rigidity because this captures the fact that price-spells are generated by firms' price-setting behavior. Since the distribution of durations is dominated by a proliferation of short contracts, the cross-sectional measure corrects for this by length-biased sampling. Modelling the price-spell durations in this way enables us to see how Taylor, Calvo and their generalizations relate to each other, and enable us to compare price-setting behavior for a given distribution of durations. We also show how the micro-data can be directly related to the macroeconomic pricing models in this setting.

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1 Introduction

Dynamic pricing and wage-setting models have become central to macroeconomic modelling in the new neoclassical synthesis approach. It has become apparent that different models of pricing have different implications for matters such as the persistence of output, prices and inflation to monetary shocks. Different models of wage or price setting imply different distributions of durations of price-spells (throughout this paper, we will use "price" as a shorthand for "wage and price"). In this paper we focus on the class of time or duration dependent models of pricing, such as Calvo and Taylor, rather than state-dependent models (Dotsey et al 1999, John and Wolman 2008). We formulate a unified framework for consistently understanding and comparing these models. We start from the idea of modelling the class of all steady state distributions of durations across a given population (in this case, the firms that set prices). In steady state there are four equivalent ways of describing this. First, there is the distribution of durations: this treats each price-spell as an individual element in the population and ignores the fact that price-spells are generated by firms (and hence there maybe links between the individual price spells). Second, there is the cross-sectional distribution of *ages*: at a point in time, how long it has been since the current price-spell began. This is like the population census. Third, we can look at the distribution in terms of *hazard rates* or *survival probabilities*: from the cross-section of ages, the probability of progressing from one age to the next one. Lastly, we can look at the cross-section of completed price-spells (lifetimes): this corresponds to the average completed price-spell across firms and hence in this context we call it the *Distribution across firms* (DAF). The main innovation of the paper is to develop a transparent framework that allows us to move between these concepts. The first three concepts (distribution of durations, cross-section of ages and hazard rates) are of course very well understood in statistics, being basic tools in demography, evolutionary biology and elsewhere. The fourth concept, the cross-sectional distribution of completed durations is more novel. However, as I argue, this concept is essential if we are to answer questions such as what is the average price-spell across firms and to apply these concepts to understand and compare different models of pricing.

Firms set prices, and if we are to measure nominal rigidity in a meaningful way, we must focus on the behavior of firms. This is a very different concept form the average length of a price-spell, a measure that is frequently used

(see for example Bils and Klenow 2004, Klenow and Kristov 2008, Nakamura and Steinsson 2008). The empirical evidence shows that there is a very wide and skewed distribution of price-spell durations, with many short price spells. Taking the average over price-spells gives an excessive weight to short durations and an underestimate of the real degree of nominal rigidity in an economy, as argued by Baharad and Eden (2004), Hoffman and Kurz-Kim (2006), Dias et al (2007). The cross-sectional approach is a form of length-biased sampling, which weights price-spells in proportion to their duration. This enables us to focus on the behavior of the firms which are generating the price-spells, and use the information that sequences of price-spells are linked because they are generated by the same firms over time. The average length of price-spell for the typical firm will have a far higher mean duration than the mean across price-spells. Furthermore, the mean cross-section duration gives the same value as the measure proposed by Baharad and Eden (2004). In the case of the French data, the mean duration of a price spell is 5.3 months, whereas the mean across firms is 14 months (Baudry et al 2007, Dixon and LeBihan 2009). With this framework, we can understand how frequency based measures of nominal rigidity (which focus on the proportion of prices changing per period) can yield quite short mean durations (around 5 months) in a way that is perfectly consistent with the average across firms being much longer (around 12 months or more). Of course, these are two measures of the same thing but from a different perspective. In this paper I argue that the cross-sectional approach is the most revealing in terms of firm behavior generating the price-spells and hence the degree of nominal rigidity.

Each of these ways of looking at the class of all steady-state distributions has a natural application to modelling price and wage setting. In the Generalized Taylor Economy (GTE) (see Kara and Dixon 2005, Taylor 1993, Coenen et al 2007), there are many sectors with different price-spell lengths, and within each sector there is a simple Taylor process. The simple Taylor economy where all contract lengths are the same is a special case of the GTE. In the Calvo approach, we have a reset probability which may be constant (as in the classical Calvo model) or duration dependent (Wolman 1999, Mash 2003 and 2004, Guerrieri 2006, Sheedy 2007, Paustian and von Hagen 2008). We show that the Calvo model with duration-dependent reset probabilities (denoted as the *Generalized Calvo model GC*) is coextensive with the set of all steady state distributions: each possible steady state age distribution has exactly one GC and one GTE which corresponds to it. Hence, using the framework, we are able to compare the different models of pricing *for*

a given distribution of durations of price spells.. This enables us to isolate the precise effect of the *pricing model* as opposed to the *differences in the distribution of durations* .

The framework in this paper also allows us to directly link microdata to models of wage and price setting. We can take a given distribution of price-spells and model it as either a GTE or a GC. We take the Bils-Klenow data set and interpret it as a Multiple Calvo (MC) economy: in each sector there is a sector-specific calvo reset probability (as in Carvalho 2006). We then generate the corresponding Hazard rate and DAF which enables us to take the three pricing models GTE, GC and MC, and compare them for *exactly the same distribution of price-spells* in a simple model economy. This enables us to highlight the differences in the pricing model controlling for the distribution of price-spells. What we find is that for this distribution at least, the three pricing models are quite close in terms of the impulse-response functions they generate in response to a monetary shock. In particular, the GC and MC are quite similar. However, there can be differences: in the particular example we find that the GTE can result in a hump shaped impulse-response for inflation, whilst the GC and MC do not.

In section 2 we review the facts about the steady state distribution of durations, ages and hazard rates. We then introduce the new concept of the distribution of durations across firms and show how all four concepts are related by simple formulae which are spreadsheet friendly. In section 3, we discuss the different measures of nominal rigidity arising form different distribution and how they relate to empirical work. In section 4, we link the concepts to different models of pricing into a general framework and see how pricing models perform in a simple macroeconomy.

2 Steady State Distributions of Durations across Firms.

We will consider the steady-state demographics of price-spells in terms of their durations. The *lifetime* of a price-spell is how long it lasts from its start to its finish, a *completed* duration. There is a continuum agents f (we will call them firms here), which set prices (or wages), represented by the unit interval $f \in [0,1]$. Time is discrete and infinite $t \in Z_+ = \{0,1,2...\infty\}$.A price event (or price-observation) is a price set by a particular firm at a

particular time: p_{ft}. A price spell is a duration, a sequence of consecutive periods that have the same price. For every price event pair $\{t, f\}$ we can assign an integer $d(t, f)$ which is the price spell duration of which the price event is part of. Furthermore, we can define the subset of reset price events, when firms set a new price:

$$R = \{(t, f) : p_{ft} \neq p_{ft-1}\} \subseteq [0, 1] \times Z_+ \tag{1}$$

The distribution of durations is derived from the set R. Let the longest duration[1] be $F < \infty$. Then we can define F subsets of R

$$R(i) = \{(t, f) \in R : d(t, f) = i\}$$

Thus $R(i)$ gives us the subset of durations of length i. The distribution of durations is simply the proportions of all durations having length $i = 1...F$:

$$\boldsymbol{\alpha}^d = \left\{\alpha_i^d\right\}_{i=1}^F \in \Delta^{F-1}$$

In steady-state this simplifies, since the distribution of durations of new price-spells is the same each period, we can take any "representative" $t > F$ and define

$$\alpha_i^d = \alpha_i^d(t) = \frac{\int_0^1 I((f, t) \in R(i))df}{\int_0^1 I((f, t) \in R)df}$$

Where I is an index function that takes the value 1 if (at our chosen t) price event (f, t) is in the relevant set, 0 otherwise. In steady-state the distribution of durations is the same as the distribution of durations taken over the subset of reset prices (new price spells).

2.1 Ages.

The age of a price-spell at time t is defined as the period of time that has elapsed since the price spell started. Formally, we can take a price event p_{ft} and define the age as:

$$\begin{aligned} A(f, t) &= 1 + \min_s [t - s] \\ s.t\ (f, s) &\in R \\ s &\leq t \end{aligned}$$

[1]The finiteness of F is merely for convenience and has no importance since it can be set arbitrarily large.

Since we have integer time, we adopt the convention that the minimum age is 1. Hence, for each (t, f) we have an associated measure of age $a(f, t)$. Let us define the subset of firms at time t that are of age $A = j$.

$$j(t) = \{f \in [0, 1] : A(f, t) = j\}$$

Then the proportion of firms aged j at t is for all $t > F$

$$\alpha_j^A = \alpha_j^A(t) = \int_0^1 I\left((f, t) \in j(t)\right) .df$$

The steady-state distribution of ages is monotonic: you cannot have more older people than younger, since to become old you must first be young. Hence the set of all possible steady state age distributions is given by:

$$\Delta_M^{F-1} = \left\{\boldsymbol{\alpha}^A \in \Delta^{F-1} : \alpha_j^A \geq 0, \alpha_j^A \geq \alpha_{j+1}^A\right\}$$

where the subscript M refers to (weak) monotonicity.

2.2 Hazard Rate.

An alternative way of looking at the steady state distribution of durations and the cross-section of ages is in terms of the hazard rate. The hazard rate at a particular age is the proportion of contracts at age i which do not last any longer (contracts which end at age i, people who die at age i). Hence the hazard rate is defined in terms of the age distribution: given the distribution of ages in steady-state $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$, the corresponding vector of hazard rates[2] $\boldsymbol{\omega} \in [0, 1]^{F-1}$ is given by[3]:

$$\omega_i = \frac{\alpha_i^A - \alpha_{i+1}^A}{\alpha_i^A}; i = 1...(F - 1) \tag{2}$$

[2]Since the maximum length is F, without loss of genersality we set $\omega_F = 1$. If $\omega_i = 1$ for some $i < F$, then i is the maximum duration and subsequent hazard rates become irrelevant. This leads to trivial non-uniqueness. We therefore define F as the *shortest* duration with a reset probability of 1.

[3]The Hazard rate can also be defined in terms of the distribution of durations.

$$\omega_i = \frac{\alpha_i^d}{\sum_{j \geq i} \alpha_j^d} = \frac{\Omega_i - \Omega_{i+1}}{\Omega_i}$$

For the relationship between continouous and discrete time used here see Kiefer (1988) and Fougere et al (2007).

Whilst it is easy to allow for an infinite series of reset probabilities less than one, we will mainly deal with the finite case where there is a final reset probability of one after F periods, although in later sections we will look at cases with infinite F.

Corresponding to the idea of a hazard rate is that of the *survival probability,* the probability at birth that the price survives for at least i periods, with $\Omega_1 = 1$ and for $i > 1$

$$\Omega_i = \Pi_{\kappa=1}^{i-1}(1 - \omega_\kappa)$$

and we define the sum of survival probabilities Σ_Ω and its reciprocal $\bar{\omega}$:

$$\Sigma_\Omega = \sum_{i=1}^{F} \Omega_i \quad \bar{\omega} = \Sigma_\Omega^{-1}$$

Clearly, we can invert (2), hence:

Observation 1 given $\boldsymbol{\omega} \in [0,1]^{F-1}$, there exists a unique corresponding age profile $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$ given by:

$$\alpha_i^A = \bar{\omega}\Omega_i \quad i = 1...F.$$

Given the flow of new contracts $\bar{\omega}$, the proportion surviving to age i is Ω_i : $\bar{\omega} = \Sigma_\Omega^{-1}$ ensures adding up. From the definition of hazard rates and Observation 1 we can move from an age distribution $\boldsymbol{\alpha}^s \in \Delta_M^{F-1}$ to the hazard profile and vice versa.[4]

Observation 2 given $\boldsymbol{\omega} \in [0,1]^{F-1}$, there exists a unique corresponding distribution of durations $\boldsymbol{\alpha}^d \in \Delta^{F-1}$ given by:

$$\alpha_i^d = \Omega_i\omega_i \quad i = 1...F.$$

The proportion of price-spells of duration i is the proportion surviving i periods and no longer. Hence there is a unique $1-1$ relationship between elements of the set of possible duration distributions and the set of possible hazard profiles.

observation 3. For any $\boldsymbol{\alpha}^d \in \Delta_M^{F-1}$, the corresponding cross-section of ages $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$ is given by

$$\alpha_i^A = \frac{\bar{\omega}}{\omega_i}\alpha_i^d$$

and vice-versa.

[4] This relationship is one of the building blocks of Life Tables (Chiang 1984), which are put to a variety of uses by demographers, actuaries and biologists.

2.3 The cross-sectional distribution of Completed Price-spells across Firms.

The steady-state age distribution $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$, distribution of durations $\boldsymbol{\alpha}^d \in \Delta^{F-1}$ or hazard profile $\boldsymbol{\omega} \in [0,1]^{F-1}$ are different ways of looking at the same object: a *panel* of price events. Each row of the panel is a trajectory of prices corresponding to a particular firm. Each column is a cross-section of all of the prices set by firms at a point in time. We now introduce a fourth distribution: it is a cross-sectional distribution of *completed* durations or lifetimes across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$. In effect, we take a representative t, and for each firm we see the completed price-spell duration at that time $d(f,t)$. If we define

$$R(i,t) = \{f \in [0,1] : d(t,f) = i\}$$

then the proportion of firms at time t with a completed duration of i, α_i is defined by:

$$\alpha_i = \alpha_i(t) = \int_0^1 I\left((f,t) \in R(i,t)\right) df$$

Under the steady-state assumption α_i is constant over time, and hence we omit the time indicator.

We can move from the distribution of ages to the distribution of completed contract lengths across firms:

Proposition 1 *Consider a steady-state age distribution $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$. There exists a unique distribution of lifetimes across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$ which corresponds to $\boldsymbol{\alpha}^A$, where*

$$\begin{aligned} \alpha_1 &= \alpha_1^A - \alpha_2^A \\ \alpha_i &= i\left(\alpha_i^A - \alpha_{i+1}^A\right) \\ &.. \\ \alpha_F &= F\alpha_F^A \end{aligned} \tag{3}$$

All proofs are in the appendix. Since there is a 1-1 mapping from age to lifetimes, we can compute the distribution of lifetimes from ages:

Corollary 1 Given a distribution of steady-state completed lifetimes across firms, $\boldsymbol{\alpha} \in \Delta^{F-1}$, there exists a unique $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$ corresponding to $\boldsymbol{\alpha}$

$$\alpha_j^A = \sum_{i=j}^{F} \frac{\alpha_i}{i} \qquad j = 1...F \tag{4}$$

The intuition behind Proposition 1 and the Corollary is clear. In a steady state, each period must look the same in terms of the distribution of ages This implies that if we look at the i period contracts, a proportion of i^{-1} must be renewed each period. Thus if we have 10 period contracts, 10% of these must come up for renewal each period. This implies that the proportion of contracts coming up for renewal each period (which have age 1) is:

$$\alpha_1^A = \sum_{i=1}^{\infty} \frac{\alpha_i}{i}$$

The proportion of contracts aged 2 is the set of contracts that were reset last period (α_1^s), less the ones that only last one periods (α_1) and so on. The set of all possible steady state distributions of durations can be characterized either by the set of all possible age distributions: $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$ or the set of all possible lifetime distributions across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$. They are just two different ways of looking at the same thing.

Proposition 1 and its corollary show that there is an exhaustive and 1-1 relationship between steady state age distributions and lifetime distributions. We can go from any age distribution and find the corresponding age distribution and vice versa. Now, since we know that there is also a 1-1 relation between Hazard rates and age distributions, we can also see that there will be a 1-1 relationship between completed contract lifetimes and hazard rates. First, we can ask what distribution of completed contract durations corresponds to a given vector of hazard rates. We can simply take observation 1 to transform the hazards into the age distribution, and then apply Proposition 1.

Corollary 2 let $\boldsymbol{\omega} \in [0,1]^{F-1}$. The distribution of lifetimes across firms corresponding to $\boldsymbol{\omega}$ is:

$$\alpha_i = \bar{\omega}.i.\omega_i.\Omega_i \colon i = 1...F \tag{5}$$

The flow of new contracts is $\alpha_1^s = \bar{\omega}$ each period. To survive for exactly i periods, you have to survive to period i which happens with probability Ω_i, and then start a new contract which happens with probability ω_i. Hence from a single cohort $\bar{\omega}.\omega_i.\Omega_i$ will have contracts that last for exactly i periods. We then sum over the i cohorts (to include all of the contracts which are in the various stages moving towards the their final period i) to get the expression.

We can also consider the reverse question: for a given distribution of completed contract lengths $\boldsymbol{\alpha}$,what is the corresponding profile of hazard rates? From Corollary 2, note that (5) is a recursive structure relating α_i and ω_i: α_i only depends on the values of ω_s for $s \leq i$.

Corollary 3 Consider a distribution of contract lengths across firms given by $\boldsymbol{\alpha} \in \Delta^{F-1}$. The corresponding hazard profile that will generate this distribution in steady state is given by $\boldsymbol{\omega} \in [0,1]^{F-1}$ where:

$$\omega_i = \frac{\alpha_i}{i}\left(\sum_{j=i}^{F}\frac{a_j}{j}\right)^{-1}$$

Corollary 4. For completeness, we can also ask for a given cross-section *DAF* $\boldsymbol{\alpha} \in \Delta^{F-1}$, what is the corresponding distribution of durations $\boldsymbol{\alpha}^d \in \Delta^{F-1}$ is:

$$\alpha_i^d = \frac{\alpha_i}{i.\bar{\omega}} \tag{6}$$

This follows directly form the comparison of (5) and observation 2. Clearly, by definition, the distribution of durations is the same as the distribution across firms resetting prices (new price-spells). The more frequent price setters (shorter price-spells) have a higher representation relative to longer price-spells. Note that the *rhs* denominator is the product of the contract length and the proportion of firms resetting price. For the values of $i < \bar{\omega}^{-1}$, the share of the duration i is greater across contracts than firms: for larger $i > \bar{\omega}^{-1}$ the share across contracts is less than the share across firms.

2.4 A Comparison of the mean duration measures.

How revealing are these measures in helping us to understand and measure nominal rigidity? Let us start by defining the three means corresponding to the three distributions:

Mean duration

$$\bar{d} = \sum_{i=1}^{F} i.\alpha_i^d = \bar{\omega}^{-1}$$

Mean age

$$\bar{A} = \sum_{i=1}^{F} i.\alpha_i^A$$

Mean duration across firms.

$$\bar{T} = \sum_{i=1}^{F} i.\alpha_i$$

There are a few simple observations that can be made. First, if we compare the two cross-sectional measures $\bar{A}$ and $\bar{T}$, we have length-biased sampling: we are more likely to observe longer price-spells[5] than in the duration measure $\bar{d}$. The reason for this is that in duration measure, we are restricting our measure to look at the start of price-spells: α_i^d is defined over the reset subset R of all price-events. In terms of the cross-section, $\bar{d}$ is a conditional mean: we are only looking at the *subset of firms who reset their price*. In the two cross-sectional measures, we are in effect selecting over *all price events in the cross-section*. Second, if we look at the age distribution, there is an *interruption* bias: the age represents an interrupted duration, only a part of the completed lifetime of the price-spell, $A(f,t) \leq d(f,t)$. Hence we have the two inequalities:

$$\begin{aligned} \bar{d} &\leq \bar{T} \\ \bar{A} &\leq \bar{T} \end{aligned}$$

where the second inequality can be made more precise: $\bar{A} = \bar{T}$ only if $F = 1$, otherwise $\bar{A} < \bar{T}$. Furthermore, since F is the longest contract for which $\alpha_F^d > 0$, we have

$$\begin{aligned} \bar{d} &= \bar{T} \text{ if } \alpha_F^d = 1 \\ \bar{d} &< \bar{T} \text{ if } \alpha_F^d < 1 \end{aligned}$$

since there can be no length-bias if all price-spells have the same duration F. If we turn to the mean age and the mean duration, there is no general

[5] If we choose a point in time at random, the probability of a price-spell being observed is proportional to its length: a 10 period spell is 10 times more likely to be observed than a 1 period contract.

inequality: we can have $\bar{A} > \bar{d}$ and $\bar{A} < \bar{d}$ depending on whether the interruption bias is larger than the length bias. Clearly, from the above we have some special cases:

$$\begin{aligned} \bar{A} &= \bar{T} = \bar{d} \text{ if } F = 1 \\ \bar{A} &< \bar{T} = \bar{d} \text{ if } F > 1 \text{ and } \alpha_F^d = 1. \end{aligned}$$

Now, for a given mean duration $\bar{d}$ there are many distribution of durations $\left\{\alpha_i^d\right\}$that generate that mean: and each such distribution will have a corresponding DAF $\{\alpha_i\}$ and mean across firms $\bar{T}$. Since we know that the mean duration is the reciprocal of the proportion of firms resetting prices $\bar{d} = \bar{\omega}^{-1}$, we can define the mapping $H(\bar{\omega}) : [0,1] \to \Delta^{F-1}$

$$H(\bar{\omega}) = \left\{ \boldsymbol{\alpha} \in \Delta^{F-1} : \sum_{i=1}^{F} \frac{\alpha_i}{i} = \bar{\omega} \right\}$$

$H(\bar{\omega})$ is the set of all $DAFs$ which are consistent with a given mean duration of price-spells $\bar{d}$ expressed in terms of the corresponding proportion of firms resetting prices $\bar{\omega}$. Clearly, since the maximum duration is F, we have $\bar{\omega} \geq F^{-1}$ so that H is non-empty[6]. Since $H(\bar{\omega})$ is defined by a linear restriction on the sector shares $\boldsymbol{\alpha}$, $H(\bar{\omega}) \subset \Delta^{F-2}$ and is closed and bounded. We can then ask what is the minimum and the maximum mean across firms consistent with a given $\bar{d}$.

The average length of price-spells across firms is $\bar{T}(\boldsymbol{\alpha}) = \sum_{i=1}^{F} i.\alpha_i$. Mathematically, we know that since $H(\bar{\omega})$ is non-empty, closed and bounded and $\bar{T}(\boldsymbol{\alpha})$ is continuous, both a maximum and a minimum will exist. Turning to the minimization problem first: we have:

$$\min \ \bar{T}(\boldsymbol{\alpha}) \quad s.t. \ \boldsymbol{\alpha} \in H(\bar{\omega}) \tag{7}$$

Proposition 2 Let $\boldsymbol{\alpha}^{\min} \in \Delta^{F-1}$ solve (7) to give the shortest average contract length $\bar{T}^{\min}$.

(a) No more than two sectors i have values greater than zero

(b) If there are two sectors $\alpha_i > 0$, $\alpha_j > 0$ then will be consecutive integers ($|i - j| = 1$).

[6] If $\bar{\omega} < F^{-1}$, then even if all price-spells were at the maximum duration F, there would be too many firms resetting prices.

(c) There is one solution iff $\bar{\omega}^{-1} = k \in Z_+$. In this case, $\alpha_k = 1$.
(d) The minimum is $\bar{T}^{\min} = \bar{\omega}^{-1} = \bar{d}$.

We can also ask what is the *maximum* average contract length consistent with a proportion of resetters $\bar{\omega}$:

$$\max \bar{T}(\boldsymbol{\alpha}) \quad s.t. \ \boldsymbol{\alpha} \in H(\bar{\omega}) \tag{8}$$

Proposition 3 Let $\boldsymbol{\alpha}^{\max} \in \Delta^{F-1}$ solve (8). Given the longest contract duration F, the distribution of contracts that maximizes the average length of contract subject to a given proportion $\bar{\omega}$ of firms resetting price

$$\begin{aligned} \alpha_F^{\max} &= \frac{F}{F-1}(1-\bar{\omega}) \\ \alpha_1^{\max} &= \frac{F}{F-1}\bar{\omega} - \frac{1}{F-1} \end{aligned}$$

with $\alpha_i^{\max} = 0$ for $i = 2...F-1$. The maximum average contract length is

$$\bar{T}^{\max} = F(1-\bar{\omega}) + 1$$

To understand Propositions 2 and 3, we just need to think of what is generating the mean duration $\bar{d}$ and the proportion of firms changing price each period $\bar{\omega}$. There is the unit interval of firms, divided into proportions with different price-spell durations $i = 1...F$. Firms with price-spell lengths i will set prices once every i^{-1} periods: the longer the price-spell, the more infrequently the firm will visit the price-setters' club. Hence, we can have the same proportion of firms re-setting price (and hence same mean duration) and increase the mean duration across firms by more longer price-spells. The maximum $T^{\max}$ is reached when we have as many F period contracts as possible, consistent with $\bar{\omega}$. In effect, this means we have a mix of 1 period and F period price-spells. The existence of a maximum relies on us assuming an upper bound F: clearly, as $F \to \infty$, $T^{\max} \to \infty$. The minimum occurs when all firms have similar price-spells: if $\bar{d}$ happens to be an integer, then all price-spells have that length and the two distributions are the same: $\boldsymbol{\alpha}^d = \boldsymbol{\alpha}$.

2.5 Examples.

In this section we provide five examples. In the first column we state the rest probabilities (hazard rates) $\{\omega_i\}$ in the second, in the second and third

the corresponding cross-sectional distributions of ages$\left\{\alpha_i^A\right\}$ and completed lifetimes across firms $\{\alpha_i\}$, and in the fourth the distribution $\left\{\alpha_i^d\right\}$ over durations. In the bottom row we compute the proportion of new contracts $\bar{\omega}$, and the means $\bar{A}, \bar{d}, \bar{T}$.

Example 1.

$$\begin{array}{llll} \omega_1 = \frac{9}{10} & \alpha_1^A = \frac{37}{40} & \alpha_1 = \frac{9}{10} & \alpha_1^d = \frac{36}{37} \\ \omega_2 = 0 & \alpha_2^A = \frac{1}{40} & \alpha_2 = 0 & \alpha_2^d = 0 \\ \omega_3 = 0 & \alpha_3^A = \frac{1}{40} & \alpha_3 = 0 & \alpha_3^d = 0 \\ \omega_4 = 1 & \alpha_4^A = \frac{1}{40} & \alpha_4 = \frac{1}{10} & \alpha_4^d = \frac{1}{37} \\ \bar{\omega} = \frac{37}{40} & \bar{A} = \frac{23}{20} & \bar{T} = \frac{13}{10} & \bar{d} = \frac{40}{37} \end{array}$$

In this example, there are two lengths of price spell: 90% of firms have 1 period spells and 10% 4 periods. Note that[7] $\bar{d} = 1.08 < \bar{A} = 1.15 < \bar{T} = 1.3$: because of the proliferation of short contracts, the mean *duration* even less than the average *age* across firms (in all the other examples, $\bar{d} > \bar{A}$). In this case the length bias outweighs the interruption bias.

Example 2.

$$\begin{array}{llll} \omega_1 = \frac{1}{4} & \alpha_1^s = \frac{32}{71} & \alpha_1 = \frac{8}{71} & \alpha_1^d = \frac{1}{4} \\ \omega_2 = \frac{1}{2} & \alpha_2^s = \frac{24}{71} & \alpha_2 = \frac{24}{71} & \alpha_2^d = \frac{3}{8} \\ \omega_3 = \frac{3}{4} & \alpha_3^s = \frac{12}{71} & \alpha_3 = \frac{27}{71} & \alpha_3^d = \frac{27}{96} \\ \omega_4 = 1 & \alpha_4^s = \frac{3}{71} & \alpha_4 = \frac{12}{71} & \alpha_4^d = \frac{3}{32} \\ \bar{\omega} = \frac{32}{71} & \bar{s} = \frac{128}{71} & \bar{T} = \frac{185}{71} & \bar{d} = \frac{71}{32} \end{array}$$

This has a rising hazard over four periods. The shares across firms and contracts both peak at period 3 with a small 4-period share. Here we have $\bar{A} = 1.80 < \bar{d} = 2.22 < \bar{T} = 2.61$.

Example 3: Simple Taylor 4.

$$\begin{array}{lll} \omega_1 = 0 & \alpha_1^s = \frac{1}{4} & \alpha_1 = \alpha_1^d = 0 \\ \omega_2 = 0 & \alpha_2^s = \frac{1}{4} & \alpha_2 = \alpha_2^d = 0 \\ \omega_3 = 0 & \alpha_3^s = \frac{1}{4} & \alpha_3 = \alpha_3^d = 0 \\ \omega_4 = 1 & \alpha_4^s = \frac{1}{4} & \alpha_4 = \alpha_4^d = 1 \\ \bar{\omega} = \frac{1}{4} & \bar{A} = \frac{5}{2} & \bar{T} = \bar{d} = 4 \end{array}$$

A simple lesson can be derived from example 4. When completed contracts have the same length, the distribution across contracts equals the

[7] decimals are to 2 places.

distribution across firms and hence has the same mean as in illustration of Proposition 2.

Example 4: Taylor's US Economy. We can now consider an example starting from an empirical distribution of completed contract lengths we can derive the corresponding GC. Taylor's US economy represents the estimated distribution of completed contract lengths[8] (in quarters) in the third column. We can represent this in terms of the hazards and the other two distributions (to 2 s.f.), distribution over contracts and the resultant averages.

$\omega_1 = 0.20$	$\alpha_1^A = 0.35$	$\alpha_1 = 0.07$	$\alpha_1^d = 0.20$
$\omega_2 = 0.34$	$\alpha_2^A = 0.28$	$\alpha_2 = 0.19$	$\alpha_2^d = 0.27$
$\omega_3 = 0.42$	$\alpha_3^A = 0.18$	$\alpha_3 = 0.23$	$\alpha_3^d = 0.18$
$\omega_4 = 0.50$	$\alpha_4^A = 0.11$	$\alpha_4 = 0.21$	$\alpha_4^d = 0.15$
$\omega_5 = 0.57$	$\alpha_5^A = 0.053$	$\alpha_5 = 0.15$	$\alpha_5^d = 0.087$
$\omega_6 = 0.58$	$\alpha_6^A = 0.023$	$\alpha_6 = 0.08$	$\alpha_6^d = 0.038$
$\omega_7 = 0.60$	$\alpha_7^A = 0.0095$	$\alpha_7 = 0.04$	$\alpha_7^d = 0.017$
$\omega_8 = 1$	$\alpha_8^A = 0.0037$	$\alpha_8 = 0.03$	$\alpha_8^d = 0.011$
$\bar{\omega} = 0.35$	$\bar{A} = 2.4$	$\bar{T} = 3.7$	$\bar{d} = 2.9$

It is interesting to note that here, unlike examples 1-3, we can really see the difference between the distribution of durations and DAF: in the duration distribution price-spells of length 1 and 2 are really boosted - we see a lot of shorter contracts. All the other durations are reduced, and in particular the longer contract lengths are much less common in the distribution across contracts and across firms. The resultant mean duration $\bar{d}$ is 77% of the mean across firms.

Example 5: Simple Calvo The Calvo model most naturally relates to the hazard rate approach to viewing the steady state distribution of durations. The simple Calvo model has a constant reset probability ω (the hazard rate) in any period that the firm will be able to review and

[8] In fact, in Taylor (1993), the ages are estimated but not reported. In Table 2.2 page 48 the second column we believe to be the distribution $\{\alpha_i\}$ although it is reported as $\{a_i^d\}$: in the text, it says that "contract lengths in the three to four quarter range appear to predominate". The third column which is reported as $\{\alpha_i\}$ is monotonic so may be ages. We have not been able to find an interpretation of Table 2.2 which is consistent with the steady state identities in this paper.

if so desired reset its price. This reset probability is exogenous and does not depend on how long the current price has been in place. The distribution of ages of price-spells is

$$\alpha_i^A = \omega\left(1-\omega\right)^{s-1} : \ s = 1...\infty$$

which has mean $\bar{A} = \sum_{i=1}^{\infty} \alpha_i^A.i = \omega^{-1}$. Applying Proposition 1(a) gives us the steady-state distribution of completed price-spells i across firms:

$$\alpha_i = \omega^2 i\left(1-\omega\right)^{i-1} : \ i = 1...\infty \tag{9}$$

which has mean $\bar{T} = 2\omega^{-1} - 1$ (see Dixon and Kara 2006)[9]. Note that for the simple Calvo model, the distribution of ages is the same as the distribution of durations: from observation 3, since $\omega = \bar{\omega}$, $\alpha_i^A = \alpha_i^d$ $i = 1...\infty$. Hence the interruption and length bias are exactly offset. We illustrate the simple Calvo model with $\omega = 0.25$, to 4 d.p.

$\omega_1 = 0.25$	$\alpha_1^A = 0.25$	$\alpha_1 = 0.0625$	$\alpha_1^d = 0.25$
$\omega_2 = 0.25$	$\alpha_2^A = 0.1875$	$\alpha_2 = 0.09375$	$\alpha_2^d = 0.1875$
$\omega_3 = 0.25$	$\alpha_3^A = 0.1406$	$\alpha_3 = 0.1055$	$\alpha_3^d = 0.1406$
$\omega_4 = 0.25$	$\alpha_4^A = 0.1052$	$\alpha_4 = 0.1055$	$\alpha_4^d = 0.1052$
$\omega_i = 0.25$	$\alpha_i^A = 0.25\left(0.75\right)^{i-1}$	$\alpha_i = \left(0.25\right)^2 i\left(0.75\right)^{i-1}$	$\alpha_i^d = \alpha_i^A$
$\bar{\omega} = 0.25$	$\bar{A} = 4$	$\bar{T} = 7$	$\bar{d} = 4$

3 How to measure price stickiness.

How should we think about nominal rigidity? Much of the recent literature has focussed on the distribution of durations: there exists a broad range of econometric and statistical methodology built up to study this (see for example Lancaster 1992). This has lead to a focus on the frequency of prices changing in a given period as an estimate of the mean duration: in effect using $\bar{\omega}$ as a basis for estimating $\bar{d}$. Conceptually, this means taking the whole population of durations and using the mean as the measure of price-stickiness. We can see that the standard comparison of the Taylor with Calvo is to equate the mean duration of price-spells: hence a 4 period Taylor

[9] Paustian and von Hagen (2008) use the mean age in cross-section to ensure comparability across their different pricing rules using this measure.

is equated with a simple Calvo model with a rest probability of $\omega = 0.25$ (see for example Kiley 2002). Since in most data sets there are a lot of short price durations, the mean duration estimated from the data seems quite small. As a ball park, in many micro-data sets derived from CPI data, the proportion of firms changing price per month is in the 20-30% range: hence the mean duration will be around 3-5 months, or $1-2$ quarters.

I want to argue that using the distribution of durations is not a good way to model what macroeconomists think of as nominal rigidity. To argue my case, I will make some thought experiments and examples. The first and most important point to make is that nominal rigidity is a result of how *firms* set prices. If we want to look at an economy and evaluate the degree of nominal rigidity, we would want to look at the behavior of firms. This is precisely how the earliest studies of price setting were undertaken: Bearl and Means (1932), Hall and Hitch (1939), Means (1935), Mills (1927) went to firms and asked them how often they changed prices (more recent examples include Blinder 1991 and 1994, Hall et al 2000). It is the behavior of firms in the economy, and their importance in terms of their share in the economy or CPI that determines nominal rigidity.

Focussing on firm behavior is essentially a cross-section perspective on price-spell durations, since at any one time, each extant price-spell is associated with one firm. We can consider the example of a world with two firms that last for two periods. One firm sets its price in both periods (single period price-spells). The other sets the price for two periods. Now if we take the firm based view, we would say that 50% of firms set $1-$period contracts, and 50% set two period contracts: the average contract is 1.5 periods. That is the approach taken in this paper. However, if we take the duration-based approach, we say that in the two periods there were 3 price-spells: two were $1-$period, and 1 was $2-$periods, so that the average duration is $1\frac{1}{3}$.

Let us take this example further: suppose we have an economy where 9 firms set prices for a year (all on January 1st), but where 1 firm sets prices τ times per year each of duration τ^{-1}. In a given year, there will be $9+\tau$ price-spells. The averages across price-spells and across firms will be (in years):

$$\begin{aligned} \bar{d}(\tau) &= \frac{9.1+\tau.\tau^{-1}}{9+\tau} = \frac{10}{9+\tau} \\ \bar{T}(\tau) &= \frac{9+\tau^{-1}}{10}. \end{aligned}$$

The economy consists of 90% firms who have rigid prices: 10% of the economy has more flexible prices. In any plausible economic model, the behavior of the 90% of firms is going to dominate: our measure of nominal rigidity should pick this up. However, we can see that as τ gets larger, more and more short price-spells are thrown up by the 1 flex price firm: in the limit as $\tau \to \infty$ we have $\bar{d}(\tau) \to 0$. As price spells in the flexible firm get shorter and shorter, they drive the mean duration to zero. This is not a plausible measure of an economy in which most prices are rigid for the whole year. The lower limit of $\bar{T}(\tau)$ is 0.9, is more reasonable. The duration based measure places an equal weight on each price-spell. However, if we want to understand the behavior of firms and the resultant behavior of the economy, we will want to place less weight on shorter spells. This is precisely what taking the cross-sectional distribution across firms does.

A second point, made by Baharad and Eden (2004), is that simply taking an average over price spells will lead to too much weight on shorter spells. Consider the following example. There is one firm. It keeps its price constant for 364 days of the year. On the 365th day it changes its price τ times with each duration τ^{-1} of a day. Now, this is not a steady-state example. However, we can calculate the mean duration price-spell (in days) as:

$$\bar{d}(\tau) = \frac{364 + \tau.\tau^{-1}}{364 + \tau} = \frac{365}{364 + \tau}$$

Again, the more frenetic the price changes in the last day, the lower the mean duration: $\bar{d}$ goes to zero as $\tau \to \infty$. Surely this is not a good measure of the price-stickiness: the price was constant for nearly all of the year, and what happens in the last day should not be able to wipe that out. Baharad and Eden propose a measure of price rigidity in which the duration of the price spell is weighted according to its duration: longer price spells occupy more of the time. The measure is the mean across time (or "per price" in their terminology):

$$BE(\tau) = \frac{364}{365}(364) + \frac{1}{365}\tau^{-1}$$

In effect, each price-spell is weighted by its duration. We can see that BE goes to 364 as $\tau \to \infty$, which is a much more accurate representation of nominal rigidity than the mean price-spell.

In both of these examples, we can see that to understand nominal rigidity, we need to place a greater weight on longer price spells. In micro-data sets, there are typically many short durations: in the CPI data, in some sectors

most prices change every month (gasoline, tomatoes and airline fares had 70% or more of prices changing per month in Bils and Klenow 2004). Using the frequency based (or direct measures) of mean duration will lead to a very misleading picture of the degree of nominal rigidity in the economy due to the proliferation of short price-spells. The mean duration of a price-spell is not a good measure of nominal rigidity because it treats each price-spell equally irrespective of length.

It turns out that both the Baharad and Eden measure BE and the mean $\bar{T}$ are equivalent in steady state[10]. Let us first define the BE measure more precisely. If we weight each duration in proportion to its length i we have the weighted distribution

$$\alpha_i^{BE} = \bar{\omega}.i\alpha_i^d$$

the RHS is multiplied by $\bar{\omega}$ to ensure adding up to unity (since $\sum_{i=1}^{F} i\alpha_i^d = \bar{\omega}^{-1}$). The sum of the weights has to add up for the average:

$$BE = \sum_{i=1}^{F} i\alpha_i^{BE} = \sum_{i=1}^{F} i(i\bar{\omega}\alpha_i^d) = \sum_{i=1}^{F} i.\alpha_i = \bar{T} \tag{10}$$

since from Corollary 4 (6) $\alpha_i = i\bar{\omega}\alpha_i^d$. Hence:

Observation 4. In steady-state $BE = \bar{T}$.

The reason that the two measures are equivalent is that they both weight the price-spells by their duration. In the cross-sectional DAF this is because of the length-biased sampling: in the BE measure, it is done directly. Much of the literature on duration analysis has tried explicitly to eliminate any length bias: if you want to find the average duration of unemployment across entrants, then looking at the average completed duration of the stock of unemployed will overestimate it (Carlson and Horrigan 1983 and Lancaster 1992[11]). In many applications you have good reason to treat each duration equally: each unemployment spell is unique and there is no link between different unemployment spells.. However, with prices, there is a panel element: price-spells across time are linked by the fact that they are

[10] This point is made by Garbriel and Reiff (2007).

[11] "picking an individual from the unemployed stock and observing his completed duration is non-randomly sampling the duration of entrants...We have in fact what is often called length-biased sampling of complete durations in which the probability that a spell will be sampled is proportional to its length" Lancaster (1992), p.95.

set by the same firm. *Focussing on the distribution of durations is in effect ignoring the panel structure and the fact that it is firms which are generating the price-spells.*

The last, but by no means least, reason that we should look at the cross-sectional measure is that this also reflects how firms and households look at things. When it sets prices, the firm's maximand is the discounted sum of future profits up to some time T (which may be infinite). Thus the weight put on a particular price spell is in a sense "proportionate" to its duration, notwithstanding the effects of discounting. Since the objective function is additive across time, a longer duration adds more items into the summation than a shorter duration. Hence firms pay attention to the flow of profits earned during price-spells roughly proportionate to their duration, given discounting. An analogous argument can be made for households and the government.

3.1 Micro data: prices are stickier than we thought.

There are now several studies using micro data: in particular the *Inflation Persistence Network* (IPN) across the Eurozone has been particularly comprehensive[12]. These studies adopt a common methodology using monthly micro CPI data across several countries which includes direct measurements in addition to the frequency based methodology, in contrast to the US studies which focus more exclusively on the frequency based estimates (Bils and Klenow (2004), Klenow and Krystov (2008), Nakamura and Steinsson (2008) and also Bunn and Ellis (2009) for the UK). Here we will consider Alvarez and Hernando (2006) for Spain (covering 1994-2003), Veronese et al (2005) for Italy (covering1996-2003), Baudry et al (2007) for France (covering 1994-2003). All these studies have data on individual products sold at individual outlets. They also have *trajectories* for prices: this is the sequence of price spells for a product at an individual outlet. We can think of each trajectory as analogous to the sequence of price contracts for an individual firm in the context of this paper. These papers all provide estimates of the average length of a price spell: both across the population of all price spells (corresponding to $\bar{d}$) and also across trajectories, where a mean duration is calculated for each trajectory and then the average is taken across trajectories (corresponding to TR). Averaging across trajectories is obviously

[12] See Dhyne et al (2006) for a summary of the IPN's findings.

related to averaging across firms, as we do in $\bar{T}$. However, as discussed in Baharad and Eden (2004) and Dixon and Le Bihan (2009), because of within trajectory diversity of price-spells, taking an unweighted average of spell durations along a trajectory will tend to overweight short spells[13], so that $TR \leq \bar{T}$[14].

There are many detailed empirical issues to do with weighting, censoring and the introduction of the Euro and sales which we can ignore here. However, we can find the direct estimates of the average duration of price spells (all durations are in months) and averages of trajectories in the three studies:

- Italy[15]: $\bar{d} = 8, TR = 13$.
- France[16]: $\bar{d} = 5.28, TR = 7.24$.
- Spain[17]: $\bar{d} = 6.2; TR = 14.7$.

In the case of Italy and Spain, averaging over trajectories leads to a considerable increase in mean duration. In the case of France, the trajectories tend to be much shorter, so this effect is weaker. However, in Dixon and Le Bihan (2009) we computed $\bar{T}$ directly with the same data set and found that $\bar{T} = 13.87$ which is more in line with Italy and Spain.

Gabriel and Reiff (2007) have developed the framework in this paper and considered the best way to estimate price stickiness. There are potentially 4 different ways of measuring price-stickiness: estimating the cross-sectional average age or completed duration, the mean duration using the reciprocal of the frequency of price-change and the hazard rate. Using a Monte-Carlo methodology, they find that in the absence of censoring and sectoral heterogeneity, all four measures are similar and "close" to the true values. However,

[13] I would like to thank Peter Gabriel and Adam Reiff for pointing this out in a comment on the ECB working paper 676 version of this paper.

[14] Also, the CPI sampling process leads to switches of outlet and product leading to incomplete trajectories, a form of panel "attrition".

[15] Veronese et al (2005), Table A2.

[16] Baudry et al (2007). Note, the estimate of $\bar{d}$ is only for unweigthed data. The trajectories in the French data only have an average length of 17 months. Hence *our* TR is taken to be their $\bar{T}^W$.

[17] Alverez and Hernando (2006). $\bar{T}$ is taken from Table 6.1 Panel C. There is no direct measure of $\bar{d}$ using the CPI weights (Panel A.gives the unweighted mean). The value quoted is derived from the inverse of the reset frequency for each sector aggregated using the CPI weights.

with the censoring of data and unobserved heterogeneity, they find that the most robust estimation methodology is the Hazard rate. In particular, the "frequency" method tends to underestimate the true mean duration: in particular, the monte carlo studies result in a downward bias in the frequency estimates of $\bar{d}$. They use micro CPI data from, Hungary 2002-6 and using the hazard-rate method, they estimate that for this data $\bar{d} = 9$ months and $\bar{T} = 16.4$ months. Hoffman and Kurz-Kim (2006) apply the Baharad and Eden weighting scheme to the German micro-CPI data (1998-2004) and find that whilst $\bar{d} = 5.3$, weighting durations[18] by length yields $\bar{T} = 26.8$.

Dias et al (2007) also consider the panel structure of the CPI and PPI data for Portugal in their estimation procedure. They suggest that for each trajectory, the price-spells are weighted by the number of spells in that trajectory (which is equivalent to weighting by the average price-spell in the trajectory), or the alternative of randomly selecting one price spell from each trajectory. Whilst these are not exactly equivalent to our length biased sampling, both mechanisms reduce the relative importance of short-spells.

4 Pricing Models with steady state distributions of durations across firms.

Having derived a unified framework for understanding the set of all possible steady state distributions of durations across firms, we can now see how this can be used to understand commonly used models of pricing behavior. Indeed, we can see how each pricing theory relates to the whole set of possible steady-state distributions.

4.1 The Generalized Taylor Economy GTE

Using the concept of the Generalized Taylor economy GTE developed in Dixon and Kara (2005), any steady-state distribution of completed durations across firms $\boldsymbol{\alpha} \in \Delta^{F-1}$can be represented by the GTE with the sector shares given by $\boldsymbol{\alpha} \in \Delta^{F-1} : \ GTE(\boldsymbol{\alpha})$. In each sector i there is an $i-$period Taylor contract, with i cohorts of equal size (since we are considering only uniform $GTEs$). The sector share is given by α_i. Since the cohorts are of equal size

[18] See Table A7, p103. The reported numbers are for price-spells unweigthed by CPI weights, for all products.

and there as many cohorts as periods, there are $\alpha_i.i^{-1}$ contracts renewed each period in sector i. This is exactly as required in a steady-state. Hence the set of all possible $GTEs$ is equivalent to the set of all possible steady-state distributions of durations. Note that the for the GTE we need to know the DAF $\boldsymbol{\alpha}$. Although we can derive the DAF from the distribution of price-spells $\boldsymbol{\alpha}^d$, the latter cannot be applied directly to any price theory. In effect, since the distribution of durations ignores the panel structure of the economy and the role of firms in setting prices, it does not directly relate to firms pricing behavior.

In a GTE, the reset price at time t in sector i x_{it} is (in log-linearised form):

$$x_{it} = \left(\frac{1}{\sum_{k=0}^{i-1} \beta^k} \right) \sum_{k=0}^{i-1} \beta^k p^*_{t+k} \tag{11}$$

where p^*_t is the optimal flex-price at time t and β the discount rate. There are F reset price equations, with $i = 1...F$. The F prices in each sector i are simply the average over the i cohorts in that sector:

$$p_{it} = \frac{1}{i} \sum_{k=0}^{i-1} x_{it-k} \tag{12}$$

The aggregate price level is simply:

$$p_t = \sum_{i=1}^{F} \alpha_i p_{it} \tag{13}$$

It is simple to verify that the age-distribution in a GTE is given by (4). If we want to know how many contracts are at aged j periods, we look at sectors with lifetimes at least as large as j, $i = j...F$. In each sector i, there is is a cohort of size $\alpha_i.i^{-1}$ which set its price j periods ago. We simply sum over all sectors $i \geq j$ to get (4). The GTE has been employed by Taylor (1993), Coenen et al (2007), Dixon and Kara (2008), Kara (2008, 2009).

4.2 The Generalized Calvo model (GC): duration dependant reset probabilities.

The Calvo model most naturally relates to the hazard rate approach to viewing the steady state distribution of durations and it has a constant hazard

rate. We now consider generalizing the Calvo model to allow for the reset probability (hazard) to vary with the age of the contract (duration dependent hazard rate). This we will denote the *Generalized Calvo Model GC*. A GC is defined by a sequence of reset probabilities: as in the previous section this can be represented by any $\boldsymbol{\omega} \in [0,1]^{F-1}$ where F is the shortest contract length with $\omega_F = 1$. From observation 1, given any possible GC there is a unique age profile $\boldsymbol{\alpha}^A \in \Delta_M^{F-1}$ corresponding to it and a unique distribution of completed contract lengths from Proposition 1. Again, from corollary 3, if we have a distribution of completed contract lengths, there is a unique GC which corresponds to it. Thus, the two approaches to modelling pricing: the GTE and the GC are comprehensive and coextensive, both being consistent with any steady-state distribution of durations[19].

The GC differs from the GTE in that when they reset prices, firms do not know how long the price-spell is going to last. There is not a sector specific reset price, but one economy wide reset price x_t with $x_{it} = x_t$ for all $i = 1...F$. The log-linearised formula for the optimal reset price at t is

$$x_t = \frac{1}{\sum_{k=1}^{F} \Omega_k \beta^{k-1}} \sum_{k=1}^{F} \Omega_k \beta^{k-1} p^*_{t+k-1} \tag{14}$$

The price in each sector i is then the average over the cohorts in that sector

$$p_{it} = \frac{1}{i} \sum_{k=0}^{i-1} x_{t-k} \tag{15}$$

with the aggregate price being given by (13) as before, where the α_i are derived from the reset probabilities $\boldsymbol{\omega} \in [0,1]^{F-1}$ using Corollary 2. The difference between the GTE and the GC lies in the whether the duration of the price-spell is known: with the GC only the distribution of price spells is known by the firm. In effect, the firm does not know ex ante which sector it is in. The GC model has been employed by Wolman (1999), Mash (2003,2004), Dotsey and King (2006), Guerrieri (2006), Sheedy (2007) and Paustian and von Hagen (2008).

[19]Note that an alternative parameterization of the duration dependent hazard rate model is to specify not the hazard rate at each duration, but rather the probability of the completed contract length at birth (see for example Guerrieri 2006).

4.3 The Multiple Calvo Model (MC).

We now use the framework to address the issue of *aggregation* over Calvo processes. Alvarez et al (2005), Bils and Klenow (2004) and Fougere et al (2007) argue the aggregate hazard rate observed in the data declines over time and that this can be attributed to the heterogeneity of hazard rates. We can define a multiple Calvo process MC as $MC(\boldsymbol{\omega}, \lambda)$ where $\boldsymbol{\omega} \in (0,1)^n$ gives a sector specific hazard rate[20] $\bar{\omega}_k$ for each sector $k = 1, ...n$ and $\boldsymbol{\lambda} \in \boldsymbol{\Delta}^{n-1}$ is the vector of shares λ_k (this might be expenditure or CPI weights). The reset price for each sector $k = 1...n$ is then:

$$x_{kt} = \frac{1}{\sum_{j=1}^{F}(1-\bar{\omega}_k)^{j-1}\beta^{j-1}} \sum_{j=1}^{F}(1-\bar{\omega}_k)^{j-1}\beta^{j-1}p^*_{t+j-1} \tag{16}$$

The average price in each sector k is then

$$p_{kt} = \sum_{j=1}^{F}(1-\bar{\omega}_k)^{j-1}\beta^{j-1}x_{kt-j+1} \tag{17}$$

And the aggregate price is then

$$p_t = \sum_{k=1}^{n}\lambda_k p_{kt} \tag{18}$$

The Multiple Calvo model has been employed by Carvalho (2006) and Carvalho and Nechio (2008) and the earlier version of this paper (2006).

4.4 The Typology of Contracts.

In terms of contract structure, we can say that the following relationships hold:

- $GC = GTE = SS$. The set of all possible steady state distributions of durations is equivalent to the set of all possible GTEs and the set of all possible GCs.

[20] The notation here should not be confused: the substrcripts k are sectoral: none of the sectoral calvo reset probabiltities are duration dependent.

- $C \subset MC \subset GC$. The set of distributions generated by the Simple Calvo is a special case of the set generated by MC which is a special case of GC.
- $ST \subset GTE = GC$ Simple Taylor is a special case of GTE, and hence also of GC.
- $ST \cap MC = \varnothing$. Simple Taylor contracts are a special case of GC, but not of MC.

Figure 1: The typology of Contracts

This is depicted in Fig 1. The GC and the GTE are coextensive, being the set of all possible steady-state distributions (Propositions 1 and corollary 3). The Simple calvo C (one reset probability) is a strict subset of the Multiple Calvo process MC which is a strict subset of the GC[21].. The simple Taylor ST and the MC are disjoint. The ST is a strict subset of the GTE. The size of the distributions is reflected by the Figure: ST has elements corresponding to the set of integers and is represented by a few dots; Calvo is represented by the unit interval; MC by the unit interval squared.

The simple Calvo and Taylor models are only applicable if there is one type of contract and no heterogeneity in the economy. If we believe the Calvo model, but that reset probabilities are heterogenous across price or wage setters, then the MC makes sense. If we do not believe the Calvo model, then either the GC or GTE are appropriate.

4.5 Price Data: an application to the Bils-Klenow Data set.

In this section, we apply our theoretical framework to the Bils-Klenow Data set (Bils and Klenow 1994). This data is the micro-price data collected monthly for the US CPI over the period 1995-7. The BK data covers 350 categories of commodities comprising 68.9% of total consumer expenditure. They focus on the proportion of prices that change in a month in each category (sector). They then derive the distribution of durations of price-spells

[21] The MC can be represented by a GC with a decreasing Hazard. See an earlier version of the paper with the same title, ECB working paper 676, Proposition 2 for a derivation in discrete time.

on the assumption that there is a sector specific Calvo reset probability in continuous time[22]. There are of course other data sets to which this method can be applied: for example Klenow and Krystov (2008) and Nakamura and Steinsson (2008) provide monthly frequency data for the US.

In this section I use the BK data to construct the distribution of durations across firms. Each sector has a sector-specific average proportion of firms resetting their price per month over the period covered. We can interpret this as a Calvo reset probability in *discrete* time. The first approach we adopt is to model this as a Multiple Calvo process $BK - MC$. The second is to model the resulting distribution across all sectors. Within each sector we have the Calvo distribution of contract lengths as derived in Dixon and Kara (2006): using the sectoral weights we can then aggregate across all sectors using the sectoral weights λ_k (the CPI weights).. This gives us the following distribution of price-spells across firms:

Fig 2: the BK Distribution of Contract lengths Across Firms

Note that the mean in our distribution is larger than is reported in BK. This is because we are looking at the mean duration across firms $\bar{T} = 4.4$ rather than mean price-spell $\bar{d} = 2.7$. With the aggregate distribution of contract lengths we can model this as either a GTE or a GC as well as an MC. We therefore have three different pricing models with exactly the same distribution of contract lengths..

4.6 Pricing Models Compared[23].

We will see how the different models of pricing differ in terms of their impulse-response using a very simple stripped-down log-linearised macro-model (see Ascari 2003). Whilst we have used an extremely simple macro-model for purposes of transparency, the pricing equations can be easily set in extended

[22]The use of continuous time leads to a lower expected expected duration at birth. If the proportion resetting price is $\bar{\omega}$, the expected duration at birth is $-1/In(1-\bar{\omega})$. This is less than the discrete time expectation $1/\bar{\omega}$. The difference gets proportinatley larger as $\bar{\omega}$ gets larger. The analysis in this paper is in discrete time becuase that is how the pricing models are employed in the literature, and it provides spreadsheet simplicity and transparency. Furthermore, a price is perfectly flexible in a quarterly model whether it changes 1 or more times a quarter.

[23]I would like to thank Engin Kara for running the simulations in dynare.

$DGSE$ models like Smets and Wouters (2003) as in Dixon and LeBihan (2009) . To model the demand side, we use the Quantity Theory[24]:

$$y_t = m_t - p_t$$

where (p_t, y_t) are aggregate price and output and m_t the money supply. We model the monetary process as $AR(1)$:

$$\begin{aligned} m_t &= m_{t-1} + \varepsilon_t \\ \varepsilon_t &= \nu\varepsilon_{t-1} + \xi_t \end{aligned}$$

where ξ_t is a white noise error term. We consider the cases of $\nu = 0$ and $\nu = 0.5$.

The optimal flexible price p_t^* at period t in all sectors is given by:

$$p_t^* = p_t + \gamma y_t$$

The key parameter γ captures the sensitivity of the flexible price to output[25]. We allow for two values of $\gamma = \{0.01, 0.2\}$: a high one and a low one as discussed in Dixon and Kara (2008).

Given this rudimentary macro-structure, we can then insert the sectoral reset-price equations[26], and sectoral price equations into the model, and aggregate according to (13) or (18).

Fig 3: Responses to a one-off monetary Shock $(\nu = 0)$

In Figure 3, we depict the responses of output, the reset price, the general price level and inflation to a one-off shock with $\gamma = 0.2$. Looking at all the graphs, it is striking that the three models of pricing have fairly similar impulse-responses: none of them are far apart. However, in all cases the MC and the GC are close together and the GTE is farther away, particularly towards the end. To understand this, we can look at the IR for the average reset price and the general price level. In the GTE case, the reset price

[24] In the case of $\nu = 0$ below, the quantity theory can be seen as resulting from the Euler equation (see Ascari 2003).

[25] This can be due to increasing marginal cost and/or an upward sloping supply curve for labour. See for example Walsh (2003) chapter 5 and Woodford (2003). chapter 3.

[26] For the GTE we have $(11, 12)$, for the GC we have $(14, 15)$, for the MC we have $(16, 17)$.

rises less on impact than the MC or GC. This reflects the greater myopia: those cohorts resetting prices look less far ahead on average, so that they do not raise prices as much as in the MC or GC case. At about 10 months however, the situation is reversed: the GTE reset price exceeds the MC and GC case: whilst the latter are slowing down price increases in anticipation of the approaching steady state, the GTE maintains momentum for longer. This comparative myopia of the GTE explains why the output response starts off above both the MC and GC, but ends up after 15 months below both.

Fig 4: Serial Correlation in Monetary growth $\nu = 0.5$

In Figure 4 we consider the autoregressive monetary policy shock and concentrate on the IR for output and inflation for both the high and the low values of γ. We find that there is now a more radical difference between the GTE and the other two models. If we look at inflation we see that there is a hump shape: the peak impact on inflation appears after the initial monetary shock: with the high value of γ it happens at 3 months: with the low value at around 20 months. Both the MC and the GC inflation responses are not hump shaped. This reflects the finding in Dixon and Kara (2008) that the Calvo model does not capture the characteristic "hump shaped" response indicated by empirical VARS. This "no hump" feature appears to be shared by its generalizations MC and GC.

This simple example of the IR of major variables shows how different models of pricing can yield different patterns of behavior even though the distribution of contract lengths are exactly the same. The MC and the GC do differ slightly, but are quite close, which reflects the fact that they have the same forward lookingness. It suggests that since the GC is computationally much simpler (you only have to model one pricing decision for all firms resetting price, rather than one for each sector), this model might be preferred to the MC.

5 Conclusions

In this paper we have developed a consistent and comprehensive framework both for analyzing different pricing models (excluding the state-dependent pricing models) and relating the pricing models to the microeconomic data. In particular, the *distribution of completed price-spells across firms* (DAF) is a key perspective which is fundamental to understanding and comparing

different models. Any steady state distribution of durations can be looked at in terms of completed durations, which suggests it can be modelled as a GTE; it can also be thought of in terms of Hazard rates which suggests the GC approach. Both the GC and the GTE are comprehensive: they can represent all possible steady states.

We argue that the economic concept of nominal rigidity is not best captured by the mean duration of a price-spell, which puts excessive weight on short spells (in the sense that all spells of any length have equal weight). The cross-sectional approach we favour is a form of length-biased sampling, which weights price-spells in proportion to their duration. The cross-sectional approach means we are averaging across the firms that set the prices and hence generate the price-spell distribution. Thus, the recent empirical evidence on the mean duration of price-spells being $4-6$ months is quite compatible with the empirical evidence from surveys that firms reset prices on average 12 months or more. This enables us to understand the different measures of nominal rigidity in the literature and why they obtain what at first seem to be different results.

As more empirical micro-data becomes available, it is vital that we adopt a framework which enables us to link the data to our macroeconomic models. Whilst the approach adopted here is limited to steady-state analysis, it does provide a consistent way for linking the micro-data to the macroeconomic models of pricing. It is for future work to see how this analysis can be applied to non-steady-state analysis and state-dependent models.

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7 Appendix.

7.1 Proof of Proposition 1 and Corollaries 1 and 3.

Proof. The proportion of firms that have a contract that last for exactly 1 period are those that are born (age 1) and do not go on to age 2. The proportion of firms that last for exactly i periods in any one cohort (born at the same time) is given by those who attain the age i but who do not make it to $i+1$: this is $\left(\alpha_i^A - \alpha_{i+1}^A\right)$ per cohort and at any time t there are i cohorts containing contracts that will last for i periods.

Clearly, since α_j^A are monotonic, $\alpha_i \leq 1$, and

$$\begin{aligned}\sum_{i=1}^{F} \alpha_i &= \sum_{i=1}^{F} i\left(\alpha_i^A - \alpha_{i+1}^A\right) \\ &= \left(\alpha_1^A - \alpha_2^A\right) + 2\left(\alpha_2^A - \alpha_3^A\right) - 3\left(\alpha_3^A - \alpha_4^A\right) .... \\ &= \sum_{i=1}^{F} \alpha_i^A = 1\end{aligned}$$

Hence $\boldsymbol{\alpha} \in \Delta^{F-1}$.

The relationship between the distribution of ages and lifetimes can be depicted in terms of matrix Algebra: in the case of $F = 4$:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \alpha_3^A \\ \alpha_4^A \end{bmatrix}$$

Clearly, the 4×4 matrix is a mapping from $\Delta^3 \to \Delta^3$: since the matrix is of full rank, the mapping from α^A to α is $1-1$. Clearly, this holds for any F. ■

7.1.1 Proof of Corollary 1:

Proof. To see this, we can rewrite (3):

$$\begin{aligned}
\alpha_1 &= \alpha_1^A - \alpha_2^A \\
\frac{\alpha_2}{2} &= \left(\alpha_2^A - \alpha_3^A\right) \\
\frac{\alpha_i}{i} &= \left(\alpha_i^A - \alpha_{i+1}^A\right) \\
\frac{\alpha_F}{F} &= \alpha_F^A
\end{aligned}$$

hence summing over all possible durations $i = 1...F$ gives

$$\sum_{i=1}^{F} \frac{\alpha_i}{i} = \sum_{i=1}^{F-1} \left(\alpha_i^A - \alpha_{i+1}^A\right) + \alpha_F^A = \alpha_1^A$$

So that by repeated substitution we get:

$$\begin{aligned}
\alpha_2^A &= \alpha_1^A - \alpha_1 = \sum_{i=2}^{F} \frac{\alpha_i}{i} \\
\alpha_j^A &= \sum_{i=j}^{F} \frac{\alpha_i}{i} \qquad j = 1...F
\end{aligned}$$

■

7.1.2 Corollary 3.

Proof. Rearranging the $F-1$ equations (5) we have:

$$\frac{\alpha_1}{\bar{\omega}} = \omega_1; \frac{\alpha_2}{2\bar{\omega}} = \omega_2 \left(1 - \omega_1\right) ... \frac{\alpha_i}{i.\bar{\omega}} = \omega_i \Omega_i; ... \frac{\alpha_F}{F\bar{\omega}} = \Omega_F$$

By repeated substitution starting from $i = 1$ we find that

$$\omega_i = \frac{\alpha_i}{i} \left(\bar{\omega} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j}\right)^{-1} \tag{19}$$

$$\Omega_i = \frac{1}{\bar{\omega}} \left[\bar{\omega} - \sum_{j=1}^{i-1} \frac{\alpha_j}{j}\right]$$

Since we know that $\omega_F = 1$, from (19)this means that:

$$1 = \frac{\alpha_F}{F}\left(\bar{\omega} - \sum_{i=1}^{F-1}\frac{\alpha_i}{i}\right)^{-1} \Rightarrow \bar{\omega} = \sum_{i=1}^{F}\frac{\alpha_i}{i}$$

Substituting the value of $\bar{\omega}$ into (19) establishes the result. ■

7.2 Proof of Proposition 2.

Proof. Firstly we will prove (a) and (b). We do this by contradiction. Let us suppose that the solution $\boldsymbol{\alpha}$ such that $\alpha_k > 0$ and $\alpha_j > 0$ and $k - j \geq 2$ We will then show that there is another feasible GTE $\boldsymbol{\alpha}'$ with $\alpha_j > 0$ and $\alpha_{j+1} > 0$ which generates a shorter average contract length.

Let us start at the proposed solution α, and in particular the two sectors k and j, whose sector shares must satisfy the two relations:

$$\begin{aligned} \alpha_k + \alpha_j &= \rho = 1 - \sum_{i=1, i\neq j,k}^{F}\alpha_i \\ \frac{\alpha_k}{k} + \frac{\alpha_j}{j} &= \eta = \bar{\omega} - \sum_{i=1, i\neq j,k}^{F}\frac{\alpha_i}{i} \end{aligned} \tag{20}$$

ρ is the total share of the two sectors: if there are only two sectors then $\rho = 1$; if there are more than two sectors with positive shares then ρ is equal 1 minus the share of the sectors other than j and k. Likewise, η is the sum of the contribution of these two sectors to $\bar{\omega}$ less the contribution of any sectors other than j and k. Note that since $k > j$,

$$\frac{\rho}{\eta} > j \tag{21}$$

We can rewrite (20) as

$$\begin{aligned} \alpha_j &= \frac{kj}{k-j}\eta - (k-j)\,\rho \\ \alpha_k &= \rho\,(1 + k - j) - \frac{kj}{k-j}\eta \end{aligned} \tag{22}$$

What we show is that we can choose $\left(\alpha'_j, \alpha_{j+1}\right)$ which satisfies the two relations above (and hence is feasible) but yields a lower average contract length.

Specifically, We choose α'_{j+1}, α'_j such that

$$\begin{aligned} \alpha'_j &= j(j+1)\eta - \rho \\ \alpha'_{j+1} - \alpha_{j+1} &= 2\rho - j(j-1)\eta \end{aligned} \tag{23}$$

Define $\Delta\alpha_{j+1} = \alpha'_{j+1} - \alpha_{j+1}$. What we are doing is redistributing the total proportion ρ over durations j and $j+1$ so that the aggregate proportion of firms resetting the price is the same: $\boldsymbol{\alpha}' \in A(\bar{\omega})$,since (23) is equivalent to (22) implies

$$\begin{aligned} \Delta\alpha_{j+1} + \alpha'_j &= \rho \\ \frac{\Delta\alpha_{j+1}}{k} + \frac{\alpha'_j}{j} &= \eta \end{aligned} \tag{24}$$

Lastly, we show that $\boldsymbol{\alpha}'$ has a lower average contract length. Since we leave the proportions of other durations constant, their contribution to the average contract length is unchanged. From (22) the contribution of durations k and j is given by

$$\begin{aligned} T_k &= ka_k + j\alpha_j \\ &= \rho\left(k + (k-j)^2\right) - kj\eta \end{aligned}$$

Likewise the contribution with α' is given by

$$\begin{aligned} T_j &= (j+1)\Delta\alpha_{j+1} + j\alpha'_j \\ &= \rho(j+2) - (j+1)j\eta \end{aligned}$$

Now we show that

$$T_k - T_{j+1} = \rho\left(k - (j+1) + (k-j)^2 - 1\right) - \eta\left(kj - (j+1)j\right)$$

Noting strict inequality (21) we have

$$\begin{aligned} T_k - T_{j+1} &> \eta\left[j\left(k - (j+1) + (k-j)^2 - 1\right) - kj + (j+1)j\right] \\ &> \eta\left[j(k-j-1)\right] > 0 \end{aligned}$$

Hence

$$\bar{T}(\boldsymbol{\alpha}) - \bar{T}(\boldsymbol{\alpha}') = T_k - T_{j+1} > 0$$

the desired contradiction.

Hence, the GTE with the minimum contract length consistent with the observed $\bar{\omega}$ cannot have strictly positive sector shares which are not consecutive integers. There are at most two strictly positive sector shares.

To prove (c) for sufficiency, if $\bar{\omega}^{-1} = k \in Z_+$, then if $\alpha_k = 1 \in A(\bar{\omega})$. If $\alpha_k < 1$ any other element of $H(\bar{\omega})$ must involve strictly positive α_j and α_i with $j - i \geq 2$, which contradicts the parts (a) and (b) of the proposition already established.

For necessity, note that if $\bar{\omega}^{-1} \notin Z_+$, then no solution with only one contract length can yield the observed proportion of firms resetting prices. ■

7.3 Proof of Proposition 3.

Proof. First, note that if the proportions are given by the equations, then the rest of the proposition follows. I know show that these equations are indeed the maximising ones. Assume the contrary, that there is a distribution $\boldsymbol{\alpha}$ with $\alpha_i > 0$ where $1 < i < F$ which gives the maximum contract length. I show that this proposed optimum can be improved upon. Hence the optimum must involve only durations $\{1, F\}$ and the given equations follow automatically. So, let us take the proposed solution, with $\alpha_i > 0$. Let us redistribute the weight on sector i between $\{1, F\}$. In order to ensure that we remain in $H(\bar{\omega})$ the additional weights must satisfy

$$\begin{aligned} \Delta\alpha_i + \frac{\Delta\alpha_F}{F} &= \frac{\alpha_i}{i} \\ \Delta\alpha_i + \Delta\alpha_F &= \alpha_i \end{aligned}$$

which gives us

$$\begin{aligned} \Delta\alpha_F &= \alpha_i \frac{F(i-1)}{i(F-1)} \\ \Delta\alpha_1 &= \alpha_i \frac{F-i}{i(F-1)} \end{aligned}$$

The resulting Change in the average contract length is

$$\begin{aligned} \Delta\bar{T} &= \alpha_i \left[\frac{F(i-1)}{i(F-1)}(F-i) - \frac{F-i}{i(F-1)}(i-1) \right] \\ &= \frac{\alpha_i (i-1)(F-1)}{i(F-1)} [F-1] > 0 \end{aligned}$$

The desired contradiction. Given that all contracts must be either 1 or F periods long, the rest of the proposition follows by simple algebra. ■

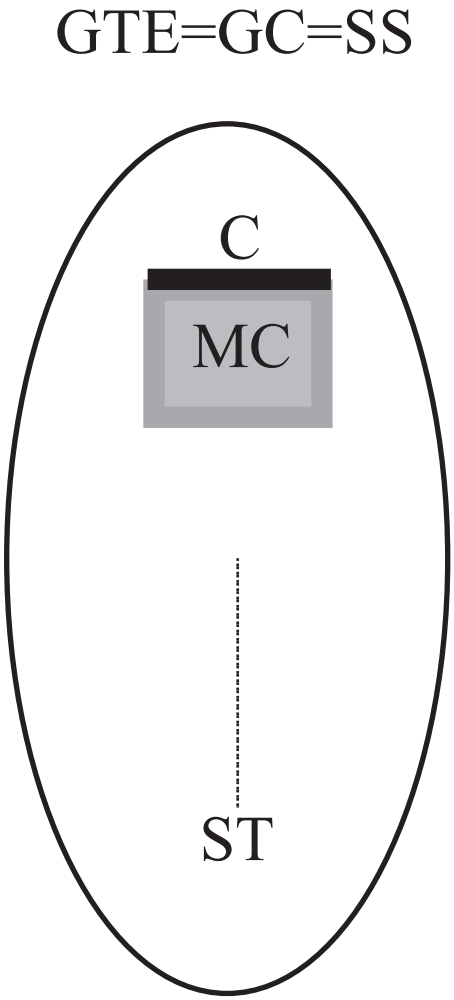

Fig.1. The Typology of contract types.

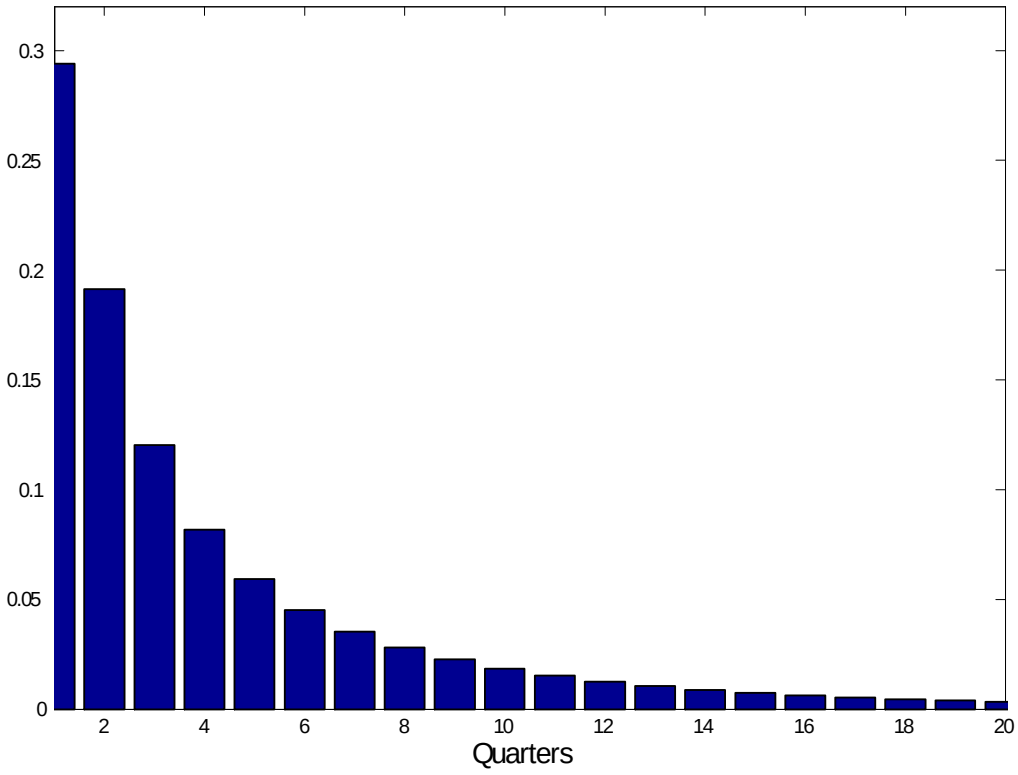

Figure 2: The BK Distribution of Contract lengths Across Firms

Figure 3: Responses to a one-off monetary Shock ($\upsilon = 0$)

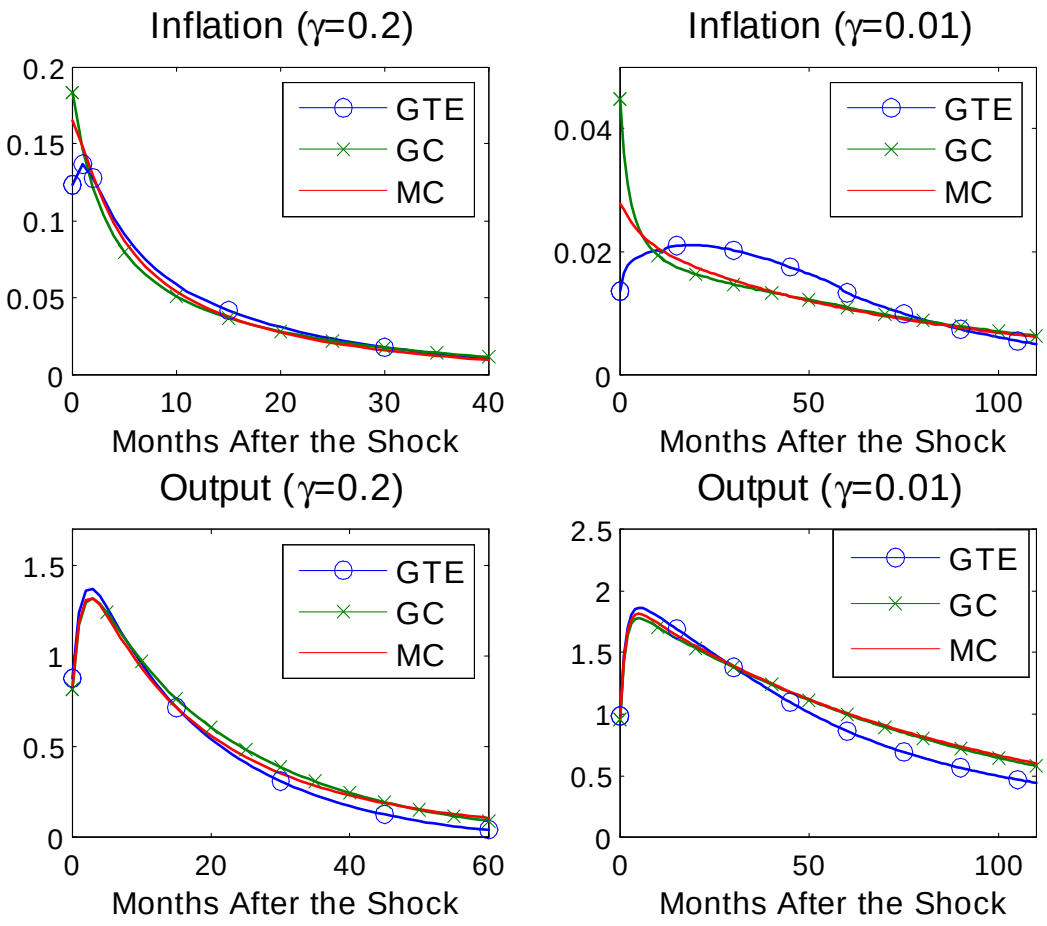

Figure 4: Responses to a one-off monetary Shock ($\upsilon = 0.5$)